CONVEX HULL OF FACE VECTORS OF COLORED COMPLEXES

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Abstract. In this paper we verify a conjecture by Kozlov (Discrete Comput Geom 18 (1997) 421–431), which describes the convex hull of the set of face vectors of r-colorable complexes on n vertices. As part of the proof we derive a generalization of Turán's graph theorem.

1. Introduction

Let Δ be a simplicial complex on n vertices and let Δ_k be the set of all faces of Δ of cardinality k. The face vector of Δ is $f(\Delta) = (n, f_2, f_3, \ldots)$ where f_k is the cardinality of Δ_k. A simplicial complex Δ is said to be r-colorable if its underlying graph (i.e., the graph with the same vertices as Δ and with edges Δ_2) is r-colorable.

Throughout this paper, by a graph G we mean a finite graph without any loops or multiple edges. The set of vertices and edges of G will be denoted by $V(G)$ and $E(G)$, respectively. The cardinality of $V(G)$ and $E(G)$ are *order* and *size* of G. A k-*clique* in G is a complete induced subgraph of G of order k. The *clique vector* of G is $c(G) = (c_1(G), c_2(G), \ldots)$, where $c_k(G)$ is the number of k-cliques in G. The *Turán graph* $T(n, r)$ is the complete r-partite graph of order n with cardinality of the maximal independent sets "as equal as possible".

A vector $g \in \mathbb{R}^d$ will be called positive if it has positive coordinates. The k-*truncation* of g, denoted by g^k, is the vector whose first k coordinates are equal to the coordinates of g, and the rest are equal to zero, for $k = 1, 2, \ldots, d$.

Kozlov conjectured [4, Conjecture 6.2] that the convex hull of the face vectors of r-colorable complexes on n vertices has a simple description in term of the clique vector of the Turán graph. The main result of this paper is to show the validity of his conjecture, more precisely:

Theorem 1.1. *The convex hull of f-vectors of r-colorable complexes on n vertices is generated by the truncations of the clique vector of Turán graph $T(n, r)$.*

The structure of the paper is as follows. In Section 2, we set up a method for finding the convex hull of the skeleta of a positive vector. The generalization of Turán's graph theorem will be proved in Section 3. Finally, in Section 4 we will prove our main result.

2. Thales' Lemma

Let $g = (g_1, \ldots, g_d)$ be a positive vector in $\mathbb{R}^d$ and denote by $\mathcal{C}_g$ the convex hull generated by the origin and all truncations of g. If $g \in \mathbb{R}^2$, then $\mathcal{C}_g$ is the boundary and interior of a right angle triangle. In this case using Thales' Intercept theorem, one can see that a positive vector (a, b) is in $\mathcal{C}_g$ if and only if $a \leq g_1$ and $(b/a) \leq (g_2/g_1)$. The following result is a generalization of this simple observation.

Lemma 2.1. *Let $g = (g_1, \ldots, g_d)$ and $f = (f_1, \ldots, f_d)$ be two positive vectors. Then $f \in \mathcal{C}_g$ if and only if $f_1 \leq g_1$ and $f_i g_j \leq f_j g_i$ for all $1 \leq j < i \leq d$.*

Proof. The vectors $g^1, \ldots, g^d$ form a basis for $\mathbb{R}^d$. So there exists $c = (c_1, \ldots, c_d) \in \mathbb{R}^d$ such that $f = \sum_1^d c_i g^i$. So we have

$$\begin{aligned} f_d &= c_d g_d, \\ f_{d-1} &= (c_{d-1} + c_d) g_{d-1}, \\ &\vdots \\ f_1 &= (c_1 + \ldots + c_d) g_1. \end{aligned}$$

On the other hand, $f \in \mathcal{C}_g$ if and only if $c_j \geq 0$ for all j and $\sum c_i \leq 1$. Therefore we have $f \in \mathcal{C}_g$ if and only if $f_1 = (\sum c_i) g_1 \leq g_1$ and $f_i g_j = (c_i + \ldots + c_d) g_i g_j \leq (c_j + \ldots + c_i + \ldots + c_d) g_j g_i = f_j f_i$. □

In the special case where g is the face vector of the $(n-1)$-dimensional simplex, the result above is already contained in the work of Kozlov [4, Section 5]. His proof, however, works in the general case as well.

3. Turán Graphs

Let us denote by $\mathcal{G}(n, r)$ the set of all graphs G of order n and clique number $\omega(G) \leq r$. Turán graph has many extremal behaviors among all graphs in $\mathcal{G}(n, r)$. Recall that *Turán graph* $T(n, r)$ is the complete r-partite graph of order n with cardinality of the maximal independent sets as equal as possible. We will denote by $t_k(n, r)$ the number of k-cliques in $T(n, r)$.

In 1941 Turán proved that among all graphs in $\mathcal{G}(n, r)$, the Turán graph $T(n, r)$ has the maximum number of edges. This result, *Turán's graph theorem*, is a cornerstone of Extremal Graph Theory. There are

many different and elegant proofs of Turán's graph theorem. Some of these proofs were discussed in [1] and in [2, Chapter 36].

Later, in 1949, Zykov generalized Turán's graph theorem by showing that $c_k(G) \leq t_k(n,r)$ for all $G \in \mathcal{G}(n,r)$ and all k. Here we state and prove a generalization of Zykov's result.

Theorem 3.1. *For any graph $G \in \mathcal{G}(n,r)$ and for each $k \in \{2,\dots,r\}$, one has*

$$\frac{c_r(G)}{t_r(n,r)} \leq \dots \leq \frac{c_k(G)}{t_k(n,r)} \leq \frac{c_{k-1}(G)}{t_{k-1}(n,r)} \leq \dots \leq \frac{c_2(G)}{t_2(n,r)} \leq 1.$$

Proof. Let $G \in \mathcal{G}(n,r)$. We may assume that G is not complete and and for a fixed k, $q_k(G) := c_k(G)/c_{k-1}(G)$ is maximum among all graphs in $\mathcal{G}(n,r)$. Let u and v be two disconnected vertices in G and define $G_{u\to v}$ to be the graph with the same vertex set as G and with edges $E(G_{u\to v}) = \big(E(G) \cup (\cup_{w\in N(v)}\{u,w\})\big) \setminus (\cup_{z\in N(u)}\{u,z\})$.
The following properties can be simply verified

- $G_{u\to v} \in \mathcal{G}(n,r)$,
- $c_k(G_{u\to v}) = c_k(G) - c_{k-1}(G[N(u)]) + c_{k-1}(G[N(v)])$.

On the other hand, it is straightforward to check that either one of $q_k(G_{u\to v})$ and $q_k(G_{v\to u})$ is strictly greater than $q_k(G)$, or they are all equal. Hence $q_k(G_{u\to v})$ is maximal.
Now consider all vertices of G that are not connected to v. let us label them by $u_1,\dots,u_m$. We define

$$G^1 := G_{u_1\to v}, \dots, G^j := G^{j-1}_{u_j\to v}, \dots, G^m := G^{m-1}_{u_m\to v}.$$

If $G^m \setminus \{v,u_1,\dots,u_m\}$ is a clique, then we stop. If not, there exists a vertex $w \in G^m \setminus \{v,u_1,\dots,u_m\}$ which is not connected to all other vertices. We repeat the above process with w and continue until the remaining vertices form a clique. So we will obtain a complete multipartite graph $H \in \mathcal{G}(n,r)$ such that $q_k(H)$ is maximum. If H is a Turán graph, then we are done. If not there exist two maximal independent sets $I_1 = \{w_1,\dots,w_m\}$ and $I_2 = \{z_1,\dots,z_l\}$ such that $m-2 \geq l$. Let H' be the graph obtained by removing all edges of the form $w_m z_i$ and adding new edges $w_m w_i$ for all $1 \leq i \leq l$. Then it is easy to see that for all j, H' has as many j-cliques as H has and, in particular $q_k(H')$ is maximum. Therefore $q_k(H'_{w_m\to z_1})$ is maximum as well and the result follows by repeating the above process.

□

Remark 3.2. The operator $G_{u\to v}$ in our proof is similar to operators used in [2, p. 238] and in [4, Theorem 3.3]. However it may belong to "folklore" graph theory, since its origin is not clear.

4. Proof of Theorem 1.1

In order to prove our main result, using Thales' Lemma 2.1, it is enough to show that for any r-colorable complex Δ on n vertices and for each k,

$$f_k(\Delta)/f_{k-1}(\Delta) \leq t_k(n,r)/t_{k-1}(n,r).$$

To prove inequalities above, we need further definitions.

Let $1 \leq k \leq r$ be fixed integers and let us denote by $\mathbb{N}_i$ the set of all positive integers whose residue modulo r is equal to i. The set of all r-colored k-subsets is

$$\mathcal{M}(k,r) = \left\{ F \in \binom{\mathbb{N}}{k} \middle| \quad |F \cap \mathbb{N}_i| \leq 1 \text{ for all } i \right\}.$$

We consider the partial order $<_p$ on $\mathcal{M}(k,r)$ defined as follows. For $T = \{t_1, \ldots, t_k\}$ and $S = \{s_1, \ldots, s_k\}$ with $t_1 < \ldots < t_k$ and $s_1 < \ldots < s_k$ in $\mathcal{M}(k,r)$, set $T <_p S$ if $t_i \leq s_i$ for every $1 \leq i \leq k$. A family $\mathcal{F} \subseteq \mathcal{M}(k,r)$ is said to be *r-color shifted* if whenever $S \in \mathcal{F}$, $T <_p S$, and $T \in \mathcal{M}(k,r)$ one has $T \in \mathcal{F}$. A simplicial complex is said to be *r-color shifted* if for any k the set of its k-faces is an r-color shifted family. It is known that for any r-colorable complex Δ on n-vertices and for any k there exists a r-color shifted complex Γ such that $f_k(\Delta) = f_k(\Gamma)$ and $f_{k-1}(\Delta) \geq f_{k-1}(\Gamma)$. (see [3, Proposition 3.1], for instance.)

Proof. We use induction on r. For $r = 1$, Δ is totally disconnected and the statement clearly holds. Now assume that the statement holds for any $(r-1)$-colorable complex. Fix a k and let Δ be an r-colorable complex on n vertices such that

$$\frac{f_k(\Delta)}{f_{k-1}(\Delta)} = \max\left\{ \frac{f_k(\Gamma)}{f_{k-1}(\Gamma)} \middle| \quad \Gamma \text{ is an } r\text{-colorable on } n \text{ vertices} \right\}.$$

We may assume that Δ is color shifted. We may also assume that for any $j \geq k$ if Δ contains the boundary of a j-simplex δ, then Δ contains δ itself. Let $I_{(1)} = \{u_1, \ldots, u_{m-1}\}$ be the set of all vertices that are not connected to the vertex 1. For $u \in I_{(1)}$ define $\Delta_{u\to 1}$ to be the complex obtained by removing all faces which contain $\{u\}$ properly and adding new faces $F \cup \{u\}$ for all $F \in \text{link}_\Delta 1$. Note that if we have an r-coloring of Δ, it is possible that u and a vertex in $\text{link}_\Delta 1$ has the same color, however we can change the color of u with the color of 1, so this construction preserves r-colorability.

It is easy to see that

$$f_j(\Delta_{u\to 1}) = f_j(\Delta) - f_{j-1}(\text{link}_\Delta u) + f_{j-1}(\text{link}_\Delta 1).$$

Hence $f_k(\Delta_{u\to 1})/f_{k-1}(\Delta_{u\to 1})$ is maximum as well. So if we define

$$\Lambda = (\dots((\Delta_{u_1\to 1})_{u_2\to 1})\dots)_{u_{m-1}\to 1},$$

then Λ is r-colorable and $f_k(\Lambda)/f_{k-1}(\Lambda)$ is maximum, since in each step our operator preserves f_k/f_{k-1} and r-colorability.

Let us denote by L and D, the subcomplex $\mathrm{link}_\Delta 1$ and the subcomplex of Δ induced by vertices of $\mathrm{link}_\Delta 1$, respectively. It is easy to see that $f_j(\Lambda) = mf_{j-1}(L) + f_j(D)$.

Claim 4.1. *$D_j = L_j$, for any $j \geq k-1$.*

Proof. It is easy to see that $L_j \subseteq D_j$. So assume that $F \in D_j$. For any $u \in F$ we have $F \setminus \{u\} \cup \{1\} \in \Delta$, by the structure of Δ. Hence the boundary of $F \cup \{1\}$ is in Δ and we have $F \cup \{1\} \in \Delta$, therefore $F \in L_j$. □

So we have

$$\frac{f_k(\Lambda)}{f_{k-1}(\Lambda)} = \frac{mf_{k-1}(L) + f_k(L)}{mf_{k-2}(L) + f_{k-1}(L)}.$$

On the other hand, since L is $(r-1)$-colorable, there exists a graph $H \in \mathcal{G}(|V(L)|, r-1)$ such that $f_t(L)/f_{t-1}(L) \leq c_t(H)/c_{t-1}(H)$ for any $2 \leq t \leq r-1$. Denote by G^k the graph obtained by joining H and a totally disconnected graph on m vertices. Clearly $G^k \in \mathcal{G}(n,r)$ and we have $c_t(G^k) = mc_{t-1}(H) + c_t(H)$ for all t. So we have

$$\begin{aligned}
c_{k-1}(G^k)f_k(\Lambda) &= (mc_{k-2}(H) + c_{k-1}(H))(mf_{k-1}(L) + f_k(L)) \\
&= m^2c_{k-2}f_{k-1}(L) + mc_{k-2}(H)f_k(L) + \\
&\quad mf_{k-1}(L)c_{k-1}(H) + c_{k-1}(H)f_k(L) \\
&\leq m^2c_{k-1}f_{k-2}(L) + mc_k(H)f_{k-2}(L) + \\
&\quad mf_{k-1}(L)c_{k-1}(H) + c_k(H)f_{k-1}(L) \\
&= c_k(G^k)f_{k-1}(\Lambda).
\end{aligned}$$

So we have proved that for any r-colorable simplicial complex on n vertices and for a fixed k there exists a graph $G^k \in \mathcal{G}(n,r)$ such that $f_k(\Delta)/f_{k-1}(\Delta) \leq c_k(G^k)/c_{k-1}(G^k)$. On the other hand by using Theorem 3.1, for all k, we have

$$\frac{c_k(G^k)}{c_{k-1}(G^k)} \leq \frac{t_k(n,r)}{t_{k-1}(n,r)},$$

as desired. □

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